

What is the other option (popular) for the end of a star?

A black hole!

This is not anything too unusual; it is just the case of excessive gravity (gravity gone wild). You can view this as a case of the limits on the escape velocity –

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

The limit is the speed of light, c .

So that means

$$c = \sqrt{\frac{2GM_{BH}}{R_{BH}}}$$

$$c^2 = \frac{2GM_{BH}}{R_{BH}}$$

$$R_{BH} = \frac{2GM_{BH}}{c^2}$$

The last of these defines the Schwarzschild radius for the black hole. You can view this as the size of the black hole, though it is technically the distance from a black hole that you can escape from so long as you are traveling at the speed of light. Anything closer is dead. And you are also dead since you can't travel at the speed of light.

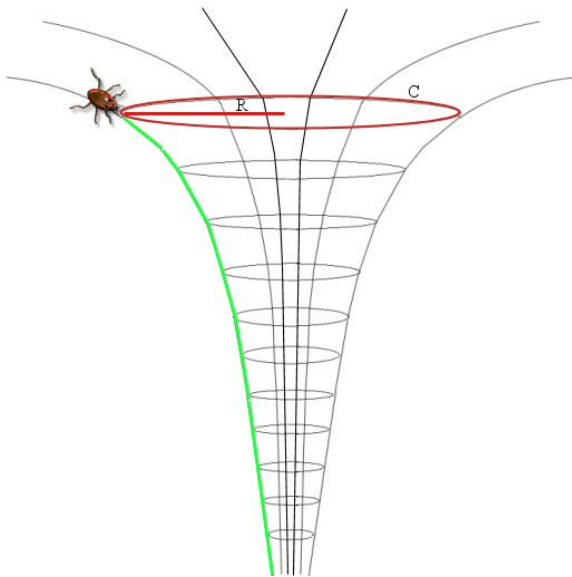
63

Another way to think of what this formula represents is as a compression threshold. If you make something that small, it will become a black hole.

So does the Schwarzschild Radius really define the “size” of a black hole, or your actual distance from a black hole?

No.

General relativity tells us that space is distorted in regions that contain mass, and the more concentrated the mass, and the greater the mass, the greater the distortion. So the space near a black hole is very warped –



Here you have a region of warped space near a black hole. The Schwarzschild radius is generally defined in Euclidean space (non-warped), so there is the belief that the black hole is only that far away (the line marked as “R”). But the actual distance is much greater due to the warping of space in this area, and is denoted by the green line in the diagram (as measured from the bug’s location). So the actual distance to the black hole isn’t really the Schwarzschild radius, so that isn’t a very useful term in helping us to determine features around the black hole.

We will instead use the circumference of

the black hole as defined by the Schwarzschild radius, since that isn't influenced by the warped space.

This value is given by

$$C_{BH} = 2\pi R_{Sch} = \frac{4\pi GM}{c^2} \quad 64$$

In order to safely investigate the black hole you don't actually want to go to the distance of the circumference (since you'll die). Instead we'll set up a location outside of the black hole and since we can't really use radius or direct measures from the black hole to define this position, we'll use the circumference of that location.

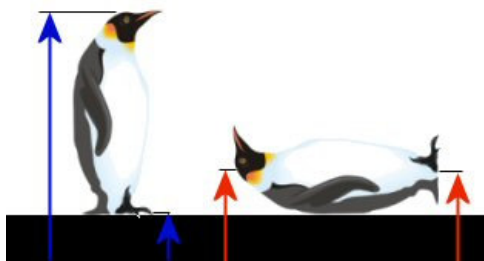
So let's define an orbital "distance" from the black hole called R_o . This would have a corresponding circumference of $C_o=2\pi R_o$. R_o is the Euclidean space distance for the orbit, and C_o is the orbital circumference. And if you want to remain alive you'll have to have $C_o > C_{bh}$.

Okay, so now you're in orbit about a black hole. That means you have to be going around the black hole at a particular pace so that you maintain that distance – not too fast or you'll fly off and not too slow or you'll fall in. You have a specific orbital period (P) that is stable.

The mass of the black hole can be measured safely from our location by using a modified version of Kepler's third law

$$M_{BH} = \frac{C_o^3}{2\pi GP^2} \quad 65$$

While you might be safe at that distance to falling into the black holes, you are also going to have to watch out for the tidal forces that are at work in this area. Tides are basically a difference in gravitational accelerations on an object, so how much is one part of you being tugged by the earth verses another part. The relative distances between those parts will help determine the tidal forces that are being experienced.



For example, the penguin on the left will feel a difference in gravitational acceleration between its head and its feet, since they are at different distances from the center of the Earth. The penguin on the right would not experience any difference in the amount of acceleration felt by its feet and head since both are at the same distance from the center of the Earth (or roughly).

The difference in gravitational acceleration due to gravity can be determined based upon the size of our orbit – since that's what we would feel in our space ship.

$$\Delta a = \frac{16\pi^3 GLM_{BH}}{C_o^3} \quad 66$$

Where the difference in acceleration, Δa , depends on the length of the object (L), the mass of the black hole and the circumference of our orbit. The length of the object should be measured radially from the black hole – sort of like the length for the penguin on the left (its height) would be the value for L . If someone were to base their orbital distance, and therefore their orbital circumference, on the size of the Schwarzschild radius (like have the orbital distance = 5xSchwarzschild radius, or 10x, or something like that), then you get the following relation –

$$\Delta a \propto \frac{M_{BH}}{C_o^3} \propto \frac{M_{BH}}{C_{BH}^3} \propto \frac{M_{BH}}{M_{BH}^3} \propto \frac{1}{M_{BH}^2}$$

Which indicates that the gravitational acceleration difference (“tidal pull”) would depend inversely upon mass of the black hole, so the largest mass black holes have the smallest tidal effects. So small mass black holes are more dangerous, while large mass black holes will not cause any serious effects. So if you have a choice, go towards a high solar mass black hole.



Not only will you get to be stretched by the tidal forces, but you are also squeezed, so that you get narrower. This is known as spaghettification. Really, that’s what it is called. The image at left shows an idealized version of this process (not to scale). Longer and narrower.



The distance of 3 Schwarzschild radii is the last stable circular orbit around a black hole. At a distance less than that, the orbits are not stable.



There are other effects on what also happens in the area of a black hole, including how light behaves. Light that is going towards a black hole will become blueshifted (wavelengths become shorter, higher energy), while light moving away from it would become redshifted (longer wavelengths, lower energy). So if you are near a black hole you’ll see light coming towards you appearing bluer by the factor



$$\lambda_{observed} = \lambda_{original} \sqrt{1 - \frac{C_{BH}}{C_o}} \quad 67a$$

And the corresponding redshift is given by

$$\lambda_{observed} = \frac{\lambda_{original}}{\sqrt{1 - \frac{C_{BH}}{C_o}}} \quad 67b$$

It is interesting to note what happens when your orbit gets close to the circumference of the black hole. In that case the stuff under the square root becomes 0, so in 67a that goes to 0 (blue shifted to 0 wavelength), while in the other it goes to infinity (infinitely long wavelengths). So if you watch light go into a black hole it gets redshifted to infinity eventually. And if you are sitting around near a black hole, you’ll see light coming in getting bluer.

And what about time?

The rules say that clocks that are in a gravitational potential (field), they run slower than if they are not in a gravitational potential. The following relation is defined for the observed passage of time for the person near the black hole by someone far from the black hole –

$$t_{near} = \frac{t_{far}}{\sqrt{1 - \frac{C_{BH}}{C_o}}} \quad 68a$$

Where t_{far} is the amount of time measured far from the black hole, and t_{near} is the time measured at the location near the black hole by the person who is far from the black hole – so it is the observed passage. So if you are far from a black hole and you have a minute pass by on your clock, the passage of time near the black hole is much less than a minute, depending on how far you are from the black hole, it could be only a few seconds. So the person outside the black hole would observe a person near the black hole to be moving very slowly, and even more slowly as they get closer to the black hole. And as you get to the Schwarzschild radius, the amount of time that passes near the black hole would be observed to be very small (since the stuff under the radical gets to be very, very small).

The above formula only works if the person near the black hole isn't actually moving around the black hole. If you are in orbit about the black hole the formula becomes

$$t_{near} = \frac{t_{far}}{\sqrt{1 - \frac{3C_{BH}}{2C_o}}} \quad 68b$$

This formula is generally better since odds are you can't just stand still near a black hole – unless you have a really good rocket to counteract it's pull.

Why do all of these peculiar things happen? It has to do with the way that space is distorted by mass and how we measure time and space (distance). Since space is warped by mass, there is an influence on the measurements of space-time due to this distortion. And of course we have to use the term space-time since space and time are interconnected – they are not separable. Generally this is the problem most people have with relativity.

Let's say you have a perspective that there is just "space". Then the difference in location between two points in your basic three dimensional universe is defined by something like

$$ds^2 = dx^2 + dy^2 + dz^2$$

Or something like that for a 3 dimensional space defined by x, y, and z. But with space and time interconnected you don't just measure "distances" between locations but space-time events between two points – or the "metric" of space-time.

$$ds^2 = -dt^2 + \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \quad 69$$

All of these have “d” since they are measures of small intervals – need that to retain accuracy. ds is not a distance but an interval in space-time, that’s why there is a time component. And the time component is negative because the event that is seen happens at a distance away (x,y,z) and in the past before you see it.

The above would only be valid if there aren’t strong distortions of space-time due to the influence of mass, so that space and time are both distorted according to the following metric – also called the Schwarzschild metric

$$ds^2 = -dt^2 \left(1 - \frac{C_{BH}}{C_o}\right) + \frac{1}{c^2} \frac{(dx^2 + dy^2 + dz^2)}{\left(1 - \frac{C_{BH}}{C_o}\right)} \quad 70$$

This basically says that whatever you measure, be it time or distance, it will be influenced by the distortion of space-time.

Other things are also influenced by the distortion of space-time, so all of those other formulae that we went over previously are now totally screwed up under the conditions of spatial distortion. For example, the formula for hydrostatic equilibrium –

$$\frac{dP}{dr} = -\rho \frac{GM}{r^2} \quad 19$$

Would be

$$\frac{dP}{dr} = -G \frac{\left(\rho + \frac{P}{c^2}\right) \left(M + \frac{4\pi r^3 P}{c^2}\right)}{r \left(r - \frac{2GM}{c^2}\right)}$$

Basically there are a lot factors of c^2 and other terms that pop up all over the place, and we don’t really want to look at all of these, do we?

Also as you get closer to the black hole, your view of the universe will change. Light is bent around by the warped space and can appear to get funneled into the black hole – and this would be light that normally wouldn’t come near you. The end result is that you see objects in more directions than you are used to seeing. In general your field of vision is around 180 degrees, but as you get closer to the black hole, the light from objects that are beyond this range starts getting funneled towards you.

You would also be forced to view the universe in only one direction. The funnel of light will get narrower and narrower as you near the black hole. But it will be difficult to see anything clearly due to the insanely large distortions that are near the black hole.

At a distance of 1.5 x Schwarzschild radius a stable orbit is only possible if you travel at the speed of light, so that means you can’t do it. This is also called the photon sphere, since light (with just the right trajectory) could remain in a stable orbit around the black hole. Anything closer than 1.5 R_{BH} would have to travel at a speed greater than light, and so that’s not possible. If you want to stay alive within the photon sphere, you’d have to

use your retro-rockets to fit the pull of gravity as well as use your rockets to orbit. Not a very fuel efficient journey.

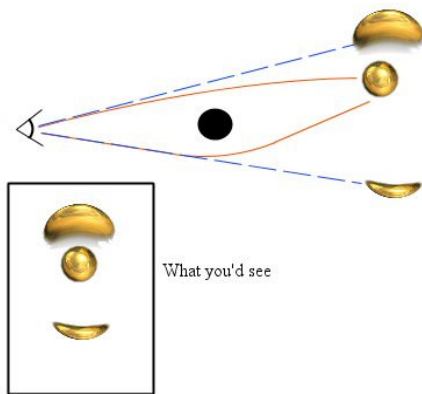
So what really happens to someone who goes into a black hole?

Let's take a trip to a 30 solar mass black hole. This is a good size since the Schwarzschild radius is about 100 km, and that's an easy number to deal with.

And let's assume that you're about 2 meters tall.

At a distance of about 100 Schwarzschild radii from the BH you have a tidal force between your head and feet equal to about 1 g. Not too bad.

As we get closer we start to notice the influence of gravitational lensing – or the distortion of images caused by the black hole. If there was a galaxy or other form located near the direction of the black hole (but behind it), the image of the galaxy would get distorted as we view it. Gravitational lensing is a very important tool used to determine the masses and distances of objects, since the form of the distortion depends upon the relative distances of the object from the observer and the mass of the lensing object (the black hole in this case). There is also the angle of alignment of the objects at play as well.



Let's say you have the arrangement pictured here – a black hole (colored black), you to the far left and a spherical object on the right. The light from that object should travel in a straight path but the black hole curves the paths, some of which are redirected towards you. In this case light that would normally not go anywhere near you gets redirected into your direction and you see two warped images of the spherical object – odds are you will not see any normal image of the sphere, I'm just including it here to sort of show its relative placement to the blobs that are also produced.

The closest you can get to the black hole and have a stable orbit is 3 Schwarzschild radii. This would also be the area that you would find any other matter falling into the black hole piling up – the inner edge of an accretion disk for example.

At a distance of 3 Schwarzschild radii, you'd have a very interesting perspective. First of all your orbit is only around 1700 km in circumference and you'd be going around at about 60% the speed of light. So one orbit would take only 0.0096 seconds (as you measure it in your spaceship). The orbital velocity is directly proportional to the black hole mass. At this distance it wouldn't really matter how fast you are moving, since the tidal forces between your head and feet is around 86,000 g.

If you get down to the Schwarzschild radius, you are at the last location that you could be seen at –though of course it would be pretty difficult to detect you since any light you give off would be redshifted so far it would be hard to pick up any light that you try to emit.

Of course in your perspective you would not see anything unusual. At this distance the tidal forces are near 1 million g. If you were free falling into the black hole (not slowing down your velocity), your velocity would be around 94% the speed of light. To the person watching you (this person is very far away) it would look like you are going at the speed of light, which also means they would not see you go any further since the time dilation has gone to infinite time for all of your regular time.

And once you pass the Schwarzschild radius, you will be dead. At the rate you are moving that will happen in a tiny fraction of a second relative to your clocks (something like 0.000197 seconds). Your friends will never see you die, since you would appear frozen at the edge of the black hole.

So nothing gets out of a black hole, right?

Maybe.

Stephen Hawking has come up with a way for something to get out of a black hole, or put another way, for a black hole to lose mass. This is via Hawking radiation. The situation requires that particles and anti-particles form near the Schwarzschild radius of the black hole. If one particle goes into the black hole and another escapes away, then as we (far from the black hole) measure the energy conservation, the particle that goes in actually has a negative energy – which would indicate a loss of mass for the black hole over time. So in a way the black hole has a specific temperature.

The relation for the temperature produced by a black hole is given by

$$T = \frac{hc^3}{16\pi^2 GMk} \quad 71a$$

Or in units of solar masses (and putting in values for all of the constants)

$$T = \frac{6.2 \times 10^{-8}}{M} \quad 71b$$

Where M is in solar masses, and temperature is in Kelvin. So a one solar mass black hole would be very cold, while an even large mass one would be even cooler. Only really small black holes are “hot”.

So if a black hole has a temperature and according to equation 63 it has a radius, that means we can define a luminosity for it – the rate at which it loses energy!

$$L = 4\pi R^2 \sigma T^4 = 4\pi \left(\frac{2GM}{c^2} \right)^2 \sigma \left(\frac{hc^3}{16\pi^2 GMk} \right)^4 = \frac{\Delta M c^2}{\Delta \tau} \quad 72$$

The last bit up there is sort of related to the Nuclear Time Scale for normal stars (equation 23), the rate at which stars give off energy and in the process lose mass. Here the mass is given by good old $E=mc^2$, and the time for mass/energy loss is $\Delta\tau$. This is basically the mass loss rate – not too different from that given for stellar winds. Okay it is really different, but it is mass loss.

Fortunately this formula can be simplified since most of it is made up of constants and we get

$$\frac{\Delta M}{\Delta \tau} = \frac{\Delta Mass}{\Delta time} = \frac{1 \times 10^{-45}}{M^2} \quad 73$$

Where the value of M is in solar masses, and the mass loss rate is in solar mass/second. So a one solar mass black hole losses mass at the rate of 1×10^{-45} solar mass/second, or around 2×10^{-15} kg/second – which isn't really anything to get excited about.

Again, the more massive the black hole, the longer it will take to lose mass. The less massive ones will evaporate much more quickly.

You can actually calculate the mass of a black hole at any given time by doing a bit of calculus on the above formula. You would end up with

$$M(t) = (M_o^3 - 1.1 \times 10^{16} t)^{1/3} \quad 74$$

Where M_o is the original mass (in kg) that you had and the time is in seconds.

You can use this formula to determine the mass at any given time, t , including the time for it to evaporate completely. But that would be a very, very long time for most “normal” black holes.

Next set of notes – the not-so-normal black holes and other weird things.