Making a star

You need to solve these formulae simultaneously for all radii in the star (model).

Conservation of Mass formula

$$\frac{dm}{dr} = 4\pi r^2 \rho \qquad 17$$

Thermal Equilibrium

$$\frac{dF}{dr} = 4\pi r^2 \rho q \qquad 18$$

Hydrostatic Equilibrium

$$\frac{dP}{dr} = -\rho \frac{GM}{r^2} \qquad 19$$

Radiative Transfer

$$\frac{dT}{dr} = -\frac{3\kappa\rho F}{4acT^3 4\pi r^2} \qquad 32$$

You can also have these formula given in terms of *dm* rather than *dr* (mass interval rather than radii intervals).

Along with these four formula you have to have some auxiliary formula

Opacity

$$\kappa = \kappa_i \rho^a T^b \qquad 31$$

Energy Production

$$q = q_0 \rho T^n \qquad 33$$

And Equation of State

$P=P_{gas}+P_{rad}=P_{e-}+P_{ions}+P_{rad}$ 24-28

where this could be ideal gases or degenerate gases.

So that is something like 7 formula for the following unknown quantities – M, T, R, F, P. Possible to do? Certainly. But these formula have to be solved for each part of the star, at the same time – they must converge to common values for M, T, R, F, P and all of those other things (κ, q, ρ).

Current method of solution is to use a grid system from the center of the star to the surface.

At center of the star you have
r=0, m=0, F=0, but T, P, are not 0!

Go out a little ways from center to distance r_1, which has corresponding values for m_1, F_1, T_1, P_1, ρ_1, κ_1, μ_1, q_1 – all of these must be calculated using the equations above.

How is this done? By using small increments of r_1, m_1, T_1, etc., such that the difference between any given interval r_2-r_1=dr, m_2-m_1=dm, T_2-T_1=dT, etc. That's where those values come in – they are differences in the parameters.

This continues all of the way from the center of the star to the surface (to level *n*) where the following conditions must be met - r_n=R, m_n=M, $F_n=L=4\pi R^2 \sigma T_{eff}^4$, $T_n=T_{eff}$ (or T_n=0), $P_n \approx 0$, $\rho_n \approx 0$.

The solutions for these problems tend to involve a lot of iterations over a grid with slight changes in values and the expected outcomes.

What does one do if you don't have a computer to solve all of this stuff? First of all you need to assume a few things –

1. Uniform composition throughout (wrong!!!) – makes pressure, opacity calculations easy, and removes evolutionary changes
2. No complex motions (convection)
3. The center/surface characteristics are set

Earliest simple models are known as Polytropic models. What does this entail?
Look at formulas 17, 18, 19 32 – there is linkage there. What if pressure were only a function of density and not temperature? Formulas 17 and19 are linked as are 18 and 32. If pressure and temperature are not linked then formula 17 and 19 can be solved by themselves since temperature doesn't play into either of these or the EOS formula.

So now you have pressure only as a function of density or

$$P = K\rho^{\gamma}$$

$$\gamma = 1 + \frac{1}{n} \qquad 35, 36$$

n is the polytropic index.
n=1.5 , γ = 5/3 → degenerate electron gas
n=3, γ=4/3 → relativistic degenerate gas
There are also situations where ideal gases can be described by this (get to this later).

Once this formula is defined for the density and pressure, it is possible to combine formulas 17 and 19 (and some calculus) to make the following messy formula

$$\frac{(n+1)K}{4\pi G n}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho \qquad 37$$

Which is (believe it or not) rather simple – all you have here is a bunch of constants, density and radius. And if you know what the conditions are in the center and the surface (r=0, R), then you can define the density throughout. And once that is defined, you can define the pressure, and then mass, then gravity.

But let's mess around with things. Let's assume that the density varies according to the relation

$$\rho = \rho_c \theta^n$$

Where ρ_c=central density, $0 \le \theta \le 1$, n=polytropic index. Basically density now is a function of the central density (a constant) and varies as you go to the surface in a certain way.

This formula can be put into 37 to make a new version of that function –

$$\left[\frac{(n+1)K}{4\pi G n \rho_c^{\frac{n-1}{n}}}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n \qquad 38$$

Which looks really ugly, but it isn't – the stuff in the [] is really just a constant of dimension length squared

$$\left[\frac{(n+1)K}{4\pi G n \rho_c^{\frac{n-1}{n}}}\right] = \alpha^2$$

But isn't r length in the star? Yes, so let's combine r and α into a new relation –

$r=\alpha\xi$

ξ is dimensionless (since r and α have dimensions), and now we can put that into formula 38, and you get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \qquad 39$$

Which is the Lane-Emden equation for a polytrope of index n.

Believe it or not, this is a really simple function to solve. Basically you define a value for n and you know what your limits are –

Center: $\theta=1$, $\xi=0$, and $d\theta/d\xi=0$

And this goes to the surface, where $R=\alpha\xi_1$

At which point the value for $\theta(\xi_1)=0$ (the surface density is 0).

Equation 39 can be solved for a variety of function of n, and these can be used to compare stellar structures.

One relation is the total mass of a polytropic star – which ends up being given by the following:

$$M = -4\pi\alpha^3\rho_c\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1} \qquad 40$$

Can also relate the central density to the average density and a polytropic constant D_n

$$\rho_c = D_n\bar{\rho} = D_n\frac{M}{\frac{4\pi}{3}R^3} \qquad 41$$

And there are also polytropic mass and radius constants (M_n and R_n), which are related via the following –

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]^n}{4\pi G} \qquad 42$$

When n=3 things get strange – mass is independent of radius is dependent only on K and M_3.
So what does that mean? You have to remember, these formulas come from the hydrostatic equilibrium and mass conservation formula (19, 17), so for various values of n, (like 3) you have hydrostatic equilibrium (a stable star), and there are unique masses that can satisfy this condition for a given value of K.
For n=1 there is another unique situation, now with the radius independent of mass and depending only on K.
And the cases in between these two $1<n<3$, you have a general relationship between mass and radius that is only a function of n.

$$R^{3-n} \propto \frac{1}{M^{n-1}}$$

So radius decreases as the mass increases – the more massive a star is, the denser it is – this is DEGENERATE!!!.

Now if you take the value of K from equation 42 and put it back into its original formula (35), you now have a relation for the central pressure, which after some manipulation becomes

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3} \qquad 43$$

Where you have another polytropic constant, B_n.

Basically you can calculate how the pressure, mass, radius, density all vary from the inside to the outside in a general approximate way by using particular values for n and known values for the constants. Here is a table of values for the polytropic constants mentioned here.

n	D_n	M_n	R_n	B_n
1.0	3.290	3.14	3.14	0.233
1.5	5.991	2.71	3.65	0.206
2.0	11.40	2.41	4.35	0.185
2.5	23.41	2.19	5.36	0.170
3.0	54.18	2.02	6.90	0.157
3.5	152.9	1.89	9.54	0.145
4.0	622.408	1.80	14.97	0.135
4.5	6189.47	1.74	31.84	0.115

There are some very interesting polytropic models, in particular those that are like electron degenerate stars (equation 26).
This is the case where n=1.5. And now we can check out what this star is like without having to do all of the icky math – really, we can since we have polytropic relations to give us information about what is happening.

First of all from equation 42 we have the situation where the mass and radius are related by

$$R \propto M^{-1.3} = 1/M^{1/3}$$

And the average density goes as

$$\bar{\rho} \propto M / R^3 \propto M^2$$

So the density increases as the square of the mass! As the mass goes up and up, the density sky rockets until eventually it reaches the point of relativistic degenerate material (n=3). And this is that special situation for mass and K. Or put another way the mass will eventually equal a constant – this is the mass limit for degenerate objects, the Chandrasekhar limit.

When all is said and done, and you substitute in the values for the polytropic values we get that the mass is simply given by

$$M_{Ch} = \frac{5.83}{\mu_e^2} \text{ solar masses} \qquad 44$$

So the Chandrasekhar mass limit isn't just one value since it will vary with composition (which changes the values for μ_e).

What else can we figure out?
Let's see if we can combine the Radiative transfer formula and hydrostatic equilibrium (32 and 19).

$$\frac{dT}{dr} = -\frac{3\kappa\rho F}{4acT^3 4\pi r^2} \quad \text{and} \quad \frac{dP}{dr} = -\rho\frac{GM}{r^2}$$

Let's divide them into one another so that we get rid of the dr term. This gives us

$$\frac{dP}{dT} = \frac{4acT^3\pi 4GM}{3\kappa F}$$

Now let's make it really simple by assuming all of the pressure is radiation pressure (makes sense because we're looking at radiation flow any ways).
In that case we use equation 28

$$P_{rad} = \frac{a}{3}T^4$$

And we have to differentiate it to get

$$\frac{dP_{rad}}{dT} = \frac{4}{3}aT^3$$

Now we have two equations with dP/dT, so let's set them equal to get……

$$F = L = \frac{4\pi cGM}{\kappa} \qquad 45$$

F=L at the surface of the star (where mass=M).
So this equation basically says if you have all of your pressure from radiation, you have to balance the energy outflow (L) with gravity (GM) – remember, this came from the hydrostatic equilibrium formula to begin with so there must be a stability in here somewhere.
This is the Eddington Luminosity formula which basically says that if a star has a luminosity greater than L it will blow itself apart – in general gravity is more powerful so it keeps the star "contained". So a star should have a luminosity less than this value to remain happy. If it doesn't, it will experience mass loss.

Eddington is also known for providing one of the first early stellar model (without a computer) which can be used to describe and predict various observed features for stars. Since he was a big shot in the game, this is also known as the "standard model".

We have at the surface the values of mass = M and energy flow=L. So somewhere inside the star we can assume that there is a relation between the mass and flux that is related to these values at the surface, or put another way

$$\frac{F}{m} = \eta \frac{L}{M}$$

At the surface η=1, and as you go into the star, η gets larger as m gets smaller. Opacity, κ, generally increases as you go outward. So what if κ and η are balanced so that

$\kappa\eta$ =constant=κ_s

Which is the surface opacity.

If you go back to how we got equation 45, it was by setting the pressure differences (dP/dT) equal to one another. What if they aren't? Why should they be equal – one is the overall pressure variation, one is the radiation pressure variation. But we can compare those two dP/dT relations and divide them into one another to get

$$\frac{dP_{rad}}{dP} = \frac{L}{4\pi cGM}\kappa_s$$

And since everything on the right is a constant, this basically becomes a relationship between the radiation pressure and the total pressure

$$P_{rad} = \frac{L}{4\pi cGM}\kappa_s P$$

What does this tell us? That the radiation pressure and the total pressure are constant from the center to the surface, that they are proportional to one another.

If the total pressure = gas pressure + radiation pressure, then we can define a value β such that

$$P_{rad} = (1-\beta)P \quad \text{and} \quad P_{gas} = \beta P$$

So that β measures the amount of radiation pressure to the overall pressure. If β =0, it is all radiation pressure, and no gas pressure. If β =1, it is all gas, and no radiation.

This can also be used with the Eddington luminosity since the luminosity approaches the Eddington value when β approaches 0.

What is the total pressure then?

$$P = \frac{P_{rad}}{(1-\beta)} = \frac{P_{gas}}{\beta}$$

$$\frac{aT^4}{3(1-\beta)} = \frac{\Re\rho T}{\beta\mu}$$

$$T = \left[\frac{3\Re(1-\beta)}{a\mu\beta}\right]^{1/3}\rho^{1/3}$$

And put the T back into the Ideal gas law to get the EOS, you'll end up with a polytrope $P = K\rho^{4/3}$ which is just like equation 42 with *n*=3, so that there is a simple relationship

between M and K. Again, mass is only dependent on a bunch of constants – but in this case one of the constants is β.
What does the mass equal?
Converting to more convenient units we get

$$M = \frac{1}{(\beta\mu)^2}\left(\frac{1-\beta}{.003}\right)^{1/2} \qquad 46$$

where M is in solar masses (this is Eddington's quartic equation). This is a funny equation since it says that there is a range of values for masses of stars – between the range of β between 0 and 1. There are several important implications for this relation.

1. β decreases as M increases – radiation pressure is very important to high mass stars
2. We can use this relation combined with the Eddington relation to derive a relationship between stellar luminosity, β, and mass, that has the form $L \propto M^3$ which is one of the possible relations for main sequence stars!
3. As the star evolves and μ changes, this will also change the influence of radiation pressure (β), such that as μ goes up, β goes down. As more metals are produced (μ goes up), the radiation pressure increases – this is especially true for supergiants and red giants.

This stuff is all rather inexact and provides a simple way of looking at how some large scale parameters change. But in the real system of stellar evolution, it is necessary to get down and dirty and solve all of the formulas that were introduced at the beginning all at once.

Stellar Stability

Models that are stable are easy – but boring.
In reality there is always some level of instability – how do you know if you have it?

Check for small levels of instability by perturbing (tweaking) parameters.
Stable – it will settle down (normal bowl)
Unstable – it will continue to grown in instability (upside down bowl)

A star could be in balance but be unstable – it just needs to be pushed.

Simple example – hydrostatic equilibrium.
Consider case with no radiation pressure, but just an ideal gas.

In that case the internal energy and the gravitational potential are balanced.

But now include the effects of radiation pressure. This will change the contribution that gravity would normally have, so that it is less (makes sense – radiation pushes outwards).

But you still have balance since a contraction of the star (increase in gravity) would cause the internal gas pressure to increase, the temperature to increase and the radiation pressure to increase – all to counteract the gravity.
And an expansion would cause a decrease in the gas pressure, a decrease in the temperature and gravity would be able to pull in easier.

Also have to consider energy balance in terms of energy production and loss, or the nuclear luminosity versus the luminosity given off.

For a happy star these should be equal. But what if L(nuclear)>L(emitted)?
What does this mean?
More energy is produced than given off – side effects are an increase in the internal pressure, temperature in the star, which causes and expansion, which causes it to cool down and that will decrease L(nuclear).
And going the other way L(emitted)>L(nuclear), a star that gives off more energy than it produces will have a will decrease its internal pressure and energy, and gravity takes advantage of that and cause contraction – heats things up and gets the nuclear rate up.

But what about degenerate material? There is no temperature dependence on that stuff – what happens in that case?

If L(nuclear)>L(emitted), the pressure and density will change but the temperature wont since it isn't linked to pressure and density! Temperature will be influenced by energy flow and since more energy is being produced than is given off, it will increase. And that will increase the L(nuclear) rate – since that depends strongly on temperature, so in that case as L(nuclear) gets bigger, it really starts to blow up. This leads to a thermonuclear run away event – this happens when there is fusion in degenerate material, and can basically cause an explosion, such as a helium flash or nova.
The resulting explosion can change things by altering the pressure and density of the material and remove the degeneracy – though that's not a guarantee.

Hydrostatic Stability – stars don't move much.
But there are cases where small changes in pressure don't die down over time but increase.
Look at adiabatic material
$P = K\rho^{\gamma}$
Change the radius, change the density, change the pressure.
For cases of $\gamma > 4/3$ there is stability
For cases for $\gamma \leq 4/3$ there is instability to some degree somewhere in there.
Does this ever happen? Yes, for relativistic electron degenerate material you have γ approaching 4/3, also where radiation pressure is dominant, γ approaches 4/3.

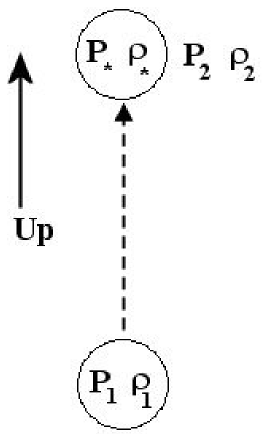
Convection

The most common form of instability. Not a fatal flaw, but can mess some stuff up.

Can influence heat flow – carries heat that would normally be carried by radiative transfer mechanisms (equation 32).
Can also lead to mixing, especially by penetrating the core and altering the composition. Mass is moved, but there is no overall mass re-structuring – the amount that goes up = the amount that goes down.

When does/doesn't convection occur?
Simple criteria – Karl Schwarzschild developed it (1906)

Start with mass Δm, P_1, ρ_1
Move it up to a region of lower pressure (P_2 and ρ_2) – what happens?
Pressure, density adjust to new values, P_*, ρ_*

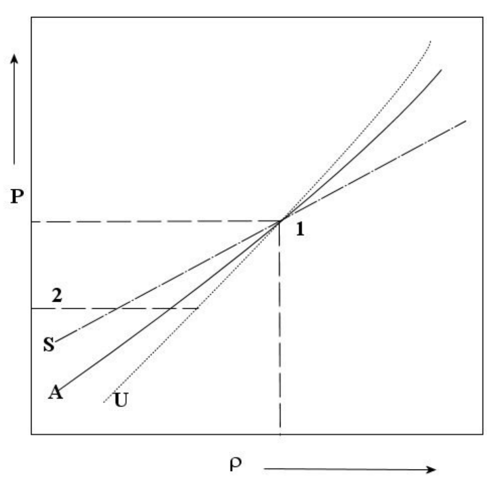


What happens next – depends upon how , P_*, ρ_* compares to values of , P_2, ρ_2

If $\rho_* > \rho_2$ it is too dense, goes back down, no convection
If $\rho_* < \rho_2$ it is too light, keeps going up, convection happens

Remember the dynamical and thermal time scales? (equations 21, 22)

In general $\tau_{thermal} >> \tau_{dyn}$
Time for the temperature to change significantly is greater than the time for the motion. Object moves without changing temperature – or exchanging energy with its environment. The object expands when it goes up, but since heat isn't changed, it's density doesn't have to equal that of the area that it is located in – and adiabatic expansion.



For an adiabatic system we have
$P = K_a \rho^{\gamma_a}$ (again)
Convection occurs if the following criteria is met –

$$\left(\frac{dP}{d\rho}\right)_* \geq \left(\frac{dP}{d\rho}\right)_a$$

Or

$$\gamma_* \geq \gamma_a$$

If < then there is no convection.
S=stable, U=unstable – slope of the line is what is above, so graphically if the slope >, convection.

So when does convection occur in a star? When does γ_a decrease in value? It goes down when ionization happens. Also high opacity values can help drive convection since that can effect the radiation pressure – provides lift.

Convection is often seen in regions where hydrogen is initially ionized (sharp change in characteristics of the material over a short distance). Happens usually around 10,000 - 20,000 K. This is also where opacity tends to peak.

Also possible to have convection in the cores of stars, usually near locations where the F is large – usually on the most massive stars have convection in the core.

When convection does happen it pretty much makes the radiative transfer laws no longer valid – (equation 32).

There is no simple way to calculate the effects of convection or to include it into models. Most method of calculation are rough approximations of the contribution of energy flow (basically taking away from the amount given in equation 32).