Whitehead, Alfred North & Bertrand Russell, "The Theory of Deduction: Primitive Ideas and Propositions", Vol. 1, 2nd edition, University Press, 1925, part I, section A, (excerpt) pp.90-94

SECTION A

THE THEORY OF DEDUCTION

THE purpose of the present section is to set forth the first stage of the deduction of pure mathematics from its logical foundations. This first stage is necessarily concerned with deduction itself, *i.e.* with the principles by which conclusions are inferred from premisses. If it is our purpose to make all our assumptions explicit, and to effect the deduction of all our other propositions from these assumptions, it is obvious that the first assumptions we need are those that are required to make deduction possible. Symbolic logic is often regarded as consisting of two coordinate parts, the theory of classes and the theory of propositions. But from our point of view these two parts are not coordinate; for in the theory of classes we deduce one proposition from another by means of principles belonging to the theory of propositions, whereas in the theory of propositions we nowhere require the theory of classes. Hence, in a deductive system, the theory of propositions necessarily precedes the theory of classes.

But the subject to be treated in what follows is not quite properly described as the theory of *propositions*. It is in fact the theory of how one proposition can be inferred from another. Now in order that one proposition may be inferred from another, it is necessary that the two should have that relation which makes the one a consequence of the other. When a proposition q is a consequence of a proposition p, we say that p *implies* q. Thus deduction depends upon the relation of *implication*, and every deductive system must contain among its premisses as many of the properties of implication as are necessary to legitimate the ordinary procedure of deduction. In the present section, certain propositions will be stated as premisses, and it will be shown that they are sufficient for all common forms of inference. It will not be shown that they are all *necessary*, and it is possible that the number of them might be diminished. All that is affirmed concerning the premisses is (1) that they are true, (2) that they are sufficient for the theory of deduction, (3) that we do not know how to diminish their number. But with regard to (2), there must always be some element of doubt, since it is hard to be sure that one never uses some principle unconsciously. The habit of being rigidly guided by formal symbolic rules is a safeguard against unconscious assumptions; but even this safeguard is not always adequate.

*1. PRIMITIVE IDEAS AND PROPOSITIONS

Since all definitions of terms are effected by means of other terms, every system of definitions which is not circular must start from a certain apparatus of undefined terms. It is to some extent optional what ideas we take as undefined in mathematics; the motives guiding our choice will be (1) to make the number of undefined ideas as small as possible, (2) as between two systems in which the number is equal, to choose the one which seems the simpler and easier. We know no way of proving that such and such a system of undefined ideas contains as few as will give such and such results*. Hence we can only say that such and such ideas are undefined in such and such a system, not that they are indefinable. Following Peano, we shall call the undefined ideas and the undemonstrated propositions *primitive* ideas and *primitive* propositions respectively. The primitive ideas are *explained* by means of descriptions intended to point out to the reader what is meant; but the explanations do not constitute definitions, because they really involve the ideas they explain.

In the present number, we shall first enumerate the primitive ideas required in this section; then we shall define *implication*; and then we shall enunciate the primitive propositions required in this section. Every definition or proposition in the work has a number, for purposes of reference. Following Peano, we use numbers having a decimal as well as an integral part, in order to be able to insert new propositions between any two. A change in the integral part of the number will be used to correspond to a new chapter. Definitions will generally have numbers whose decimal part is less than ·1, and will be usually put at the beginning of chapters. In references, the integral parts of the numbers of propositions will be distinguished by being preceded by a star; thus "*1·01" will mean the definition or proposition so numbered, and "*1" will mean the chapter in which propositions have numbers whose integral part is 1, *i.e.* the present chapter. Chapters will generally be called "numbers."

PRIMITIVE IDEAS.

(1) *Elementary propositions.* By an "elementary" proposition we mean one which does not involve any variables, or, in other language, one which does not involve such words as "all," "some," "the" or equivalents for such words. A proposition such as "this is red," where "this" is something given in sensation, will be elementary. Any combination of given elementary propositions by means of negation, disjunction or conjunction (see below) will

* The recognized methods of proving independence are not applicable, without reserve, to fundamentals. Cf. *Principles of Mathematics*, § 17. What is there said concerning primitive propositions applies with even greater force to primitive ideas.

be elementary. In the primitive propositions of the present number, and therefore in the deductions from these primitive propositions in ∗2—∗5, the letters p, q, r, s will be used to denote elementary propositions.

(2) *Elementary propositional functions.* By an "elementary propositional function" we shall mean an expression containing an undetermined constituent, *i.e.* a variable, or several such constituents, and such that, when the undetermined constituent or constituents are determined, *i.e.* when values are assigned to the variable or variables, the resulting value of the expression in question is an elementary proposition. Thus if p is an undetermined elementary proposition, "not-p" is an elementary propositional function.

We shall show in ∗9 how to extend the results of this and the following numbers (∗1—∗5) to propositions which are not elementary.

(3) *Assertion.* Any proposition may be either asserted or merely considered. If I say "Caesar died," I assert the proposition "Caesar died," if I say "'Caesar died' is a proposition," I make a different assertion, and "Caesar died" is no longer asserted, but merely considered. Similarly in a hypothetical proposition, *e.g.* "if $a = b$, then $b = a$," we have two unasserted propositions, namely "$a = b$" and "$b = a$," while what *is* asserted is that the first of these implies the second. In language, we indicate when a proposition is merely considered by "*if* so-and-so" or "*that* so-and-so" or merely by inverted commas. In symbols, if p is a proposition, p by itself will stand for the unasserted proposition, while the asserted proposition will be designated by

$$\text{“}\vdash . p.\text{”}$$

The sign "$\vdash$" is called the assertion-sign*; it may be read "it is true that" (although philosophically this is not exactly what it means). The dots after the assertion-sign indicate its range; that is to say, everything following is asserted until we reach either an equal number of dots preceding a sign of implication or the end of the sentence. Thus "$\vdash : p . \supset . q$" means "it is true that p implies q," whereas "$\vdash . p . \supset \vdash . q$" means "$p$ is true; therefore q is true†." The first of these does not necessarily involve the truth either of p or of q, while the second involves the truth of both.

(4) *Assertion of a propositional function.* Besides the assertion of definite propositions, we need what we shall call "assertion of a propositional function." The general notion of asserting *any* propositional function is not used until ∗9, but we use at once the notion of asserting various special *elementary* propositional functions. Let ϕx be a propositional function whose argument is x; then we may assert ϕx without assigning a value to x. This is done, for example, when the law of identity is asserted in the form "A is A." Here A is left undetermined, because, however A may be determined, the result will be true. Thus when we assert ϕx, leaving x undetermined, we are asserting an ambiguous value of our function. This is only legitimate if, however the ambiguity may be determined, the result will be true. Thus take, as an illustration, the primitive proposition ∗1·2 below, namely

$$\text{“}\vdash : p \vee p . \supset . p,\text{”}$$

i.e. "'p or p' implies p." Here p may be *any* elementary proposition: by leaving p undetermined, we obtain an assertion which can be applied to any particular elementary proposition. Such assertions are like the particular enunciations in Euclid: when it is said "let ABC be an isosceles triangle; then the angles at the base will be equal," what is said applies to *any* isosceles triangle; it is stated concerning *one* triangle, but not concerning a definite one. All the assertions in the present work, with a very few exceptions, assert propositional functions, not definite propositions.

As a matter of fact, no constant elementary proposition will occur in the present work, or can occur in any work which employs only logical ideas. The ideas and propositions of logic are all *general*: an assertion (for example) which is true of Socrates but not of Plato, will not belong to logic*, and if an assertion which is true of both is to occur in logic, it must not be made concerning either, but concerning a variable x. In order to obtain, in logic, a definite proposition instead of a propositional function, it is necessary to take some propositional function and assert that it is true always or sometimes, *i.e.* with all possible values of the variable or with some possible value. Thus, giving the name "individual" to whatever there is that is neither a proposition nor a function, the proposition "every individual is identical with itself" or the proposition "there are individuals" will be a proposition belonging to logic. But these propositions are not elementary.

(5) *Negation.* If p is any proposition, the proposition "not-p," or "p is false," will be represented by "$\sim p$." For the present, p must be an *elementary* proposition.

(6) *Disjunction.* If p and q are any propositions, the proposition "p or q," *i.e.* "either p is true or q is true," where the alternatives are to be not mutually exclusive, will be represented by

$$\text{“}p \vee q.\text{”}$$

This is called the *disjunction* or the *logical sum* of p and q. Thus "$\sim p \vee q$" will mean "p is false or q is true"; "$\sim (p \vee q)$" will mean "it is false that either p or q is true," which is equivalent to "p and q are both false"; "$\sim (\sim p \vee \sim q)$" will mean "it is false that either p is false or q is false," which is equivalent to "p and q are both true"; and so on. For the present, p and q must be elementary propositions.

* We have adopted both the idea and the symbol of assertion from Frege.

† Cf. *Principles of Mathematics*, § 38.

* When we say that a proposition "belongs to logic," we mean that it can be expressed in terms of the primitive ideas of logic. We do not mean that logic *applies* to it, for that would of course be true of any proposition.

The above are all the primitive ideas required in the theory of deduction. Other primitive ideas will be introduced in Section B.

Definition of Implication. When a proposition q follows from a proposition p, so that if p is true, q must also be true, we say that *p implies q.* The idea of implication, in the form in which we require it, can be defined. The meaning to be given to implication in what follows may at first sight appear somewhat artificial; but although there are other legitimate meanings, the one here adopted is very much more convenient for our purposes than any of its rivals. The essential property that we require of implication is this: "What is implied by a true proposition is true." It is in virtue of this property that implication yields proofs. But this property by no means determines whether anything, and if so what, is implied by a false proposition. What it does determine is that, if p implies q, then it cannot be the case that p is true and q is false, *i.e.* it must be the case that either p is false or q is true. The most convenient interpretation of implication is to say, conversely, that if either p is false or q is true, then "p implies q" is to be true. Hence "p implies q" is to be defined to mean: "Either p is false or q is true." Hence we put:

***1·01.** $p \supset q . = . \sim p \vee q$ Df.

Here the letters "Df" stand for "definition." They and the sign of equality together are to be regarded as forming one symbol, standing for "is defined to mean*." Whatever comes to the left of the sign of equality is defined to mean the same as what comes to the right of it. Definition is not among the primitive ideas, because definitions are concerned solely with the symbolism, not with what is symbolised; they are introduced for practical convenience, and are theoretically unnecessary.

In virtue of the above definition, when "$p \supset q$" holds, then either p is false or q is true; hence if p is true, q must be true. Thus the above definition preserves the essential characteristic of implication; it gives, in fact, the most general meaning compatible with the preservation of this characteristic.

PRIMITIVE PROPOSITIONS.

***1·1.** Anything implied by a true elementary proposition is true. Pp†.

The above principle will be extended in *9 to propositions which are not elementary. It is not the same as "*if* p is true, then *if* p implies q, q is true." This is a true proposition, but it holds equally when p is not true and when p does not imply q. It does not, like the principle we are concerned with, enable us to assert q simply, without any hypothesis. We cannot express the principle symbolically, partly because any symbolism in which p is variable only gives the *hypothesis* that p is true, not the *fact* that it is true‡.

* The sign of equality not followed by the letters "Df" will have a different meaning, to be defined later.

† The letters "Pp" stand for "primitive proposition," as with Peano.

‡ For further remarks on this principle, cf. *Principles of Mathematics*, § 38.