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Basic Laws of Arithmetic
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1. PRIMITIVE SIGNS

i. INTRODUCTION: FUNCTION, CONCEPT, RELATION[9]

§1. The function is unsaturated.

If we are asked to state the original meaning of the word "function" as used in mathematics, it is easy to fall into calling function of x an expression, formed from "x" and particular numbers by use of the notation for sum, product, power, difference, and so on. This is incorrect, because a function is here represented as an *expression*, as a concatenation of signs, not as what is designated thereby. Hence one will attempt to say, in place of "expression", rather "denotation of an expression". But now, there occurs in the expression the letter "x", which does not denote a number as the sign "2" does, for example, but only indeterminately indicates one. For different numerals that we put in the place of "x" we obtain in general different denotations. For example, if for "x" in the expression

$$\text{``}(2 + 3x^2)x\text{''}$$

we substitute the numerals "0", "1", "2", "3" in order, then we obtain as corresponding denotations the numbers 0, 5, 28, 87. None of these denotations can claim to be our function. The essence of the function manifests itself rather in the connection it establishes between the numbers whose signs we put for "x" and the numbers that then appear as denotations of our expression—a connection intuitively represented in the course of the

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[9] Cf. my lecture *Über Function und Begriff* (Jena, 1891) and my essay "Über Begriff und Gegenstand", in the *Vierteljahrsschrift für wissenschaftliche Philosophie,* Vol. XVI, no. 2 (1892). My *Begriffsschrift* (Halle, 1879) no longer fully corresponds to my present standpoint, and hence should be used only with caution to elucidate that set forth here.

curve whose equation in rectangular coördinates is

$$"y = (2 + 3x^2)x".$$

Accordingly the essence of the *function* lies in that part of the expression which is there over and above the "x". The expression for a *function* is *in need of completion, unsaturated.* The letter "x" serves only to hold places open for a numeral that is 6
to complete the expression, and in this way renders recognizable the particular type of need for completion that constitutes the specific nature of the function designated above. Hereafter, the letter "ξ" will be used for this purpose instead of "x".[10] This holding-open is to be understood as follows: all places at which "ξ" stands must be filled always by the same sign, never by different ones. I call these places *argument-places*, and that whose sign (name) occupies these places in a given case, I call the *argument* of the function for this case. The function is completed by the argument; what it becomes on completion I call the *value* of the function for the argument. Thus we obtain a name of the value of a function for an argument, if we fill the argument-places in the name of the function with the name of the argument. In this way, for example, "$(2+3\cdot1^2)\cdot1$" is a name of the number 5, composed of the function-name "$(2+3\xi^2)\xi$" and "1". Thus the argument is not to be counted a part of the *function,* but serves to complete the function, which in itself is *unsaturated.* In the sequel, where use is made of an expression like "the function $\Phi(\xi)$", it is always to be observed that "ξ" contributes to the designation of the function only so far as it renders recognizable the argument-places, but not in such a way that the essence of the function is altered if some sign is substituted for "ξ".

§2. Truth-values. Denotation and sense. Thought. Object.

To the fundamental arithmetical operations mathematicians have added, as constituting functions, the process of proceeding to a limit in its various forms, as infinite series, differential quotients, and integrals; and finally have understood the word "function" so widely that in some cases the connection between argument and value of the function can no longer be designated by the signs of mathematical analysis, but only by words. Another extension has been to admit complex numbers as arguments and consequently as values of functions. In both directions I have gone still farther. That is, while on the one hand the signs of analysis have not hitherto always been sufficient, on the other hand not all of them have been employed in forming function-names, in that "$\xi^2 = 4$" and "$\xi > 2$", for example, were not allowed to count as names of functions—as I allow them to do. But this is also to say that the domain of the values of functions cannot remain restricted to numbers; for if I take as arguments of the function $\xi^2 = 4$ the numbers 0, 1, 2, 3 in order, I do not obtain numbers [as values]. The expressions

$$"0^2 = 4", "1^2 = 4", "2^2 = 4", "3^2 = 4"$$

are expressions some of true, some of false thoughts. I put this
7 as follows: the value of the function $\xi^2 = 4$ is either the *truth-value* of what is true or that of what is false.[11] It can be seen from this that I do not mean to assert anything if I merely write down an equation, but that I merely *designate* a truth-value, just as I do not assert anything if I merely write down "2^2", but merely *designate* a number. I say: the *names* "$2^2 = 4$" and "$3 > 2$" *denote* the same truth-value, which I call for short *the True*. Likewise, for me "$3^2 = 4$" and "$1 > 2$" *denote* the same truth-value, which I call for short *the False*, precisely as the name "2^2" *denotes* the number four. Accordingly I call the number four the *denotation* of "4" and of "2^2", and I call the True the denotation of "$3 > 2$". However, I distinguish from the *denotation* of a name its *sense*. "2^2" and "$2 + 2$" do not have the same *sense,* nor do "$2^2 = 4$" and "$2 + 2 = 4$" have the same *sense*. The sense of a name of a truth-value I call a *thought*. I further say a name *expresses* its sense and *denotes* its denotation. I *designate* with the name that which it denotes.

Thus the function $\xi^2 = 4$ can have only two values, namely the True for the arguments 2 and -2, and the False for all other arguments.

The domain of what is admitted as argument must also be extended to objects in general. *Objects* stand opposed to functions. Accordingly I count as *objects* everything that is not a

[10] However, nothing is here stipulated for the Begriffsschrift. Rather, the "ξ" will not occur at all in the developments of the Begriffsschrift itself; I shall use it only in the exposition of it, and in elucidations.

[11] I have justified this more thoroughly in my essay "Über Sinn und Bedeutung" in the *Zeitschrift für Philosophie und philosophische Kritik,* 100 (1892).

function, for example, numbers, truth-values, and the courses-of-values to be introduced below. The names of objects—the *proper names*—therefore carry no argument-places; they are saturated, like the objects themselves.

§3. Course-of-values of a function. Concept. Extension of a concept.

I use the words

"the function $\Phi(\xi)$ has the same *course-of-values* as the function $\Psi(\xi)$"

generally to denote the same as the words

"the functions $\Phi(\xi)$ and $\Psi(\xi)$ have always the same value for the same argument".

We have this circumstance with the functions $\xi^2 = 4$ and $3\xi^2 = 12$, at least if numbers are taken as arguments. However, we can imagine the signs for squaring and multiplication to be so defined that the function

$$(\xi^2 = 4) = (3\xi^2 = 12)$$

has the True as value for every argument whatever. At this point we may also use an expression from logic: "the concept *square root of 4* has the same extension as the concept *some-* 8
thing whose square trebled is 12". With such functions, whose value is always a truth-value, one may accordingly say, instead of "course-of-values of the function", rather "extension of the concept"; and it seems appropriate to call directly a *concept* a function whose value is always a truth-value.

§4. Functions of two arguments.

Hitherto I have spoken only of functions of a single argument; but we can easily pass on to *functions of two arguments*. These are *doubly in need of completion*, in the sense that a function of one argument is obtained once a completion by means of one argument has been effected. Only by means of yet another completion do we attain an object, and this is then called the *value* of the function for the two arguments. Just as the letter "ξ" served us with functions of one argument, so here we make use of the letters "ξ" and "ζ" to indicate the twofold unsaturatedness of functions of two arguments, as in

$$\text{``}(\xi + \zeta)^2 + \zeta\text{''}.$$

By substituting (for example) "1" for "ζ", we saturate the function in such a way that in $(\xi + 1)^2 + 1$ we still have a function,

but of one argument. This way of using the letters "ξ" and "ζ" must always be kept in mind if an expression occurs like "the function $\Phi(\xi,\zeta)$" (cf. n. 10, above). I call the places at which "ξ" stands *ξ-argument-places*, and those at which "ζ" stands *ζ-argument-places*. I say that the ξ-argument-places are *related* to one another, and likewise for the ζ-argument-places; while I call a ξ-argument-place *not related* to a ζ-argument-place.

The functions of two arguments $\xi = \zeta$ and $\xi > \zeta$ always have a truth-value as value (at least if the signs "=" and ">" are appropriately defined). Such functions it will be appropriate to call *relations*. In the first relation, for example, 1 stands to 1, and in general every object to itself; in the second, for example, 2 stands to 1. We say that the object Γ *stands to* the object Δ *in the relation* $\Psi(\xi,\zeta)$ if $\Psi(\Gamma,\Delta)$ is the True. Likewise we say that the object Δ *falls under* the concept $\Phi(\xi)$ if $\Phi(\Delta)$ is the True. Of course it is presupposed in this that the functions $\Phi(\xi)$ and $\Psi(\xi,\zeta)$ always have as value a truth-value.[12]

9 *ii. SIGNS FOR FUNCTIONS*

§5. Judgment and thought. Judgment-stroke and horizontal.

We have already said that in a mere equation there is as yet no assertion; "2 + 3 = 5" only designates a truth-value, without its being said which of the two it is. Again, if I wrote

$$\text{``}(2 + 3 = 5) = (2 = 2)\text{''}$$

and presupposed that we knew 2 = 2 to be the True, I still should not have asserted thereby that the sum of 2 and 3 is 5; rather I should only have designated the truth-value of *"2 + 3 = 5"'s denoting the same as "2 = 2"*. We therefore require another special sign to be able to assert something as true. For this

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[12] There is a difficulty here which can easily obscure the true state of affairs and hence arouse suspicion as to the correctness of my view. If we compare the expression "the truth-value of *Δ's falling under the concept* $\Phi(\xi)$" with "$\Phi(\Delta)$", we see that what really corresponds to the expression "$\Phi(\ )$" is

"the truth-value of *( )'s falling under the concept* $\Phi(\xi)$",

and not

"the concept $\Phi(\xi)$".

These last words therefore do not really designate a concept (in our sense), even though by their linguistic form it appears as if they do. As to the awkward position in which language here finds itself, cf. my essay "Über Begriff und Gegenstand".

purpose I let the sign "⊢" precede the name of the truth-value, so that for example in

"⊢$2^2 = 4$",[13]

it is asserted that the square of 2 is 4. I distinguish the *judgment* from the *thought* in this way: by a *judgment* I understand the acknowledgement of the truth of a *thought*. The presentation in Begriffsschrift of a judgment by use of the sign "⊢" I call a *proposition of Begriffsschrift* or briefly a *proposition*. I regard this "⊢" as composed of the vertical line, which I call the *judgment-stroke*, and the horizontal line, which I will now simply call the *horizontal*.[14] The horizontal will mostly occur fused with other signs, as here with the judgment-stroke, and thereby will be protected against confusion with the *minus sign*. Where it does occur apart, for purposes of distinction it must be made somewhat longer than the minus sign. I regard it as a function-name, as follows:

——Δ

is the True if Δ is the True; on the other hand it is the False if Δ is not the True.[15] Accordingly,

——ξ

is a function whose value is always a truth-value—or by our stipulation, a concept. Under this concept there falls the True and only the True. Thus, 10

"——$2^2 = 4$"

[13] I frequently make use here, in a provisional way, of the notations for the sum, product, power, although these signs have here not yet been defined, to enable me to form examples more easily and to facilitate understanding by means of hints. But we must keep it in mind that nothing is made to rest on the denotations of these notations.

[14] I used to call it the *content-stroke*, when I still combined under the expression "possible content of judgment" what I have now learned to distinguish as truth-value and thought. Cf. my essay "Über Sinn und Bedeutung."

[15] Obviously the sign "Δ" may not be denotationless, but must denote an object. Denotationless names must not occur in the Begriffsschrift. The stipulation above is made in such a way that "——Δ" denotes something under all circumstances so long merely as "Δ" denotes something. Otherwise ——ξ would not be a concept having sharp boundaries, thus in our sense not a concept at all. I here use *capital Greek letters* as if they were names denoting something, although I do not specify their denotation. In the developments of the Begriffsschrift itself they will occur no more than will "ξ" and "ζ".

denotes the same thing as "$2^2 = 4$", namely the True. In order to dispense with brackets I specify that everything standing to the right of the horizontal is to be regarded as a whole that occupies the argument-place of the function-name "——ξ", except as *brackets* prohibit this.

"——$2^2 = 5$"

denotes the False, thus the same thing as does "$2^2 = 5$"; as against this,

"——2"

denotes the False, thus something different from the number 2. If Δ is a truth-value, then ——Δ is the same truth-value, and consequently

$\Delta =$ (——Δ)

is the True. But this is the False if Δ is not a truth-value. We can therefore say that

$\Delta =$ (——Δ)

is the truth-value of Δ*'s being a truth-value*.

Accordingly the function ——$\Phi(\xi)$ is a concept and the function ——$\Psi(\xi,\zeta)$ is a relation, regardless of whether $\Phi(\xi)$ is a concept or $\Psi(\xi,\zeta)$ a relation.

Of the two signs of which "⊢" is composed, only the judgment-stroke contains the act of assertion.

§6. Negation-stroke. Amalgamation of horizontals.

We need no special sign to declare a truth-value to be the False, so long as we possess a sign by which either truth-value is changed into the other; it is also indispensable on other grounds. I now stipulate:

The value of the function

┬ξ

shall be the False for every argument for which the value of the function

——ξ

is the True; and shall be the True for all other arguments.

Accordingly we possess in

┬ξ

a function whose value is always a truth-value; it is a concept, under which falls every object with the sole exception of the True. From this it follows that "┬Δ" always denotes the same thing as "┬(——Δ)", and as "——┬Δ", and as "——┬(——Δ)". Hence we regard "┬" as composed of the